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A NOTE ON THE FORM OF THE SHEAR COEFFICIENT

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Abstract—In an earlier paper, it was noted that the expressions derived for the shear stiffness coefficients for various cross-sectional shapes all had a common general form. It was found that Poisson's ratio always appeared to be encapsulated within a simple rational function. This led to the conjecture that this would prove to be true in general. New results obtained by different means have supported this conjecture, and have lead to a re-examination of the theory, yielding a proof which is valid for simply-connected cross-sections. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Earlier work by the author (1991) on the shear stiffness of prismatic beams was based on the concept of a characteristic response to shear. Using Saint-Venant's principle, it is possible to show that if a shear force is applied to one end of a prismatic beam, the response to it decays along the beam towards a unique linear variation. The strain energy associated with this variation is composed of the bending strain energy and the shear strain energy. The shear stiffness of the section can be determined from this expression for the shear strain energy.

An analysis of the response of a prismatic beam to shear is given by Love (1952) for example. This method is rather cumbersome and a simpler and more elegant approach developed by Timoshenko (1922) and described in Timoshenko and Goodier (1970) is generally preferred. The expressions used for the bending and shear stresses induced at a distance z along a beam from an end shear force S are

$$\sigma_{zz} = \frac{Szx}{I}; \quad \tau_{xz} = \frac{\partial \phi}{\partial y} - \frac{Sx^2}{2I} + f(y); \quad \tau_{yz} = -\frac{\partial \phi}{\partial x}. \tag{1}$$

Here, x and y are the principal axes for the cross-section, S acts in the x direction and I is the second moment of area of the cross-section about the y axis. These equations automatically satisfy the equations of equilibrium for arbitrary functions $\phi(x, y)$ and $f(y)$. The compatibility conditions are satisfied if

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{v}{1+v}\frac{Sy}{I} - \frac{\mathrm{d}f}{\mathrm{d}y} \tag{2}$$

where v is Poisson's ratio. The condition of zero surface tractions on the boundary of the cross-section is satisfied if

$$\frac{\partial \phi}{\partial s} = \left[\frac{Sx^2}{2I} - f(y)\right]\frac{\mathrm{d}y}{\mathrm{d}s} = 0. \tag{3}$$

Then on the boundary of a simply-connected cross-section, ϕ can be taken as zero without loss of generality. The function $f(y)$ is chosen so that when $\mathrm{d}y/\mathrm{d}s$ is not zero on the boundary, the contents of the square brackets are zero. Excluding the cases where the boundary itself is taken to be a function of v, it follows that $f(y)$ is not a function of Poisson's ratio. From (1), the shear strain energy per unit length is given by

$$U_S = \frac{1}{2G}\int_A (\tau_{xz}^2+\tau_{yz}^2)\,\mathrm{d}A$$

$$= \frac{1}{2G}\int_A \left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]+\left[-\frac{Sx^2}{2I}+f\right]\left[2\frac{\partial\phi}{\partial y}-\frac{Sx^2}{2I}+f\right]\mathrm{d}A \tag{4}$$

where A is the area of the cross-section. From Stoke's theorem,

$$\int_A \left[\frac{\partial}{\partial x}\left(\phi\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\phi\frac{\partial\phi}{\partial y}\right)\right]\mathrm{d}A = \int_C \phi\left[-\frac{\partial\phi}{\partial y}\mathrm{d}x+\frac{\partial\phi}{\partial x}\mathrm{d}y\right] = 0 \tag{5}$$

where the second integral is around the contour of the section and is zero because ϕ is zero on the boundary. It then follows from (2) and (5) that

$$\int_A \left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]\mathrm{d}A = -\int_A \phi\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)\mathrm{d}A = \int_A \phi\left(\frac{\mathrm{d}f}{\mathrm{d}y}-\frac{\nu}{1+\nu}\frac{Sy}{I}\right)\mathrm{d}A. \tag{6}$$

Similarly, it follows from Stoke's theorem that

$$\int_A 2\frac{\partial}{\partial y}\left[\phi\left(f-\frac{Sx^2}{2I}\right)\right]\mathrm{d}A = -\int_C 2\phi\left(f-\frac{Sx^2}{2I}\right)\mathrm{d}x = 0$$

$$\text{or}\quad \int_A 2\frac{\partial\phi}{\partial y}\left(f-\frac{Sx^2}{2I}\right)\mathrm{d}A = -\int_A 2\phi\frac{\mathrm{d}f}{\mathrm{d}y}\mathrm{d}A. \tag{7}$$

From (6) and (7), (4) now becomes

$$U_S = \frac{1}{2G}\int_A \left(f-\frac{Sx^2}{2I}\right)^2-\phi\left(\frac{\mathrm{d}f}{\mathrm{d}y}+\frac{\nu}{1+\nu}\frac{Sy}{I}\right)\mathrm{d}A. \tag{8}$$

2. FORM OF THE SHEAR COEFFICIENT

Suppose that

$$\phi = \phi_0+\frac{\nu}{1+\nu}\phi_1 \tag{9}$$

where ϕ_0 is the solution for the case when Poisson's ratio is zero. As ϕ_0 and ϕ are zero on the boundary of the section, ϕ_1 will also be zero. From (2), the conditions on these new functions within the cross-sectional area are

$$\frac{\partial^2\phi_0}{\partial x^2}+\frac{\partial^2\phi_0}{\partial y^2} = -\frac{\mathrm{d}f}{\mathrm{d}y},\quad \frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial y^2} = \frac{Sy}{I}. \tag{10}$$

Then, neither of these are functions of Poisson's ratio. Now

$$-\int_A \phi\left(\frac{\mathrm{d}f}{\mathrm{d}y}+\frac{\nu}{1+\nu}\frac{Sy}{I}\right)\mathrm{d}A$$

$$= -\int_A \left(\phi_0+\frac{\nu}{1+\nu}\phi_1\right)\left[-\frac{\partial^2\phi_0}{\partial x^2}-\frac{\partial^2\phi_0}{\partial y^2}+\frac{\nu}{1+\nu}\left(\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial y^2}\right)\right]\mathrm{d}A. \tag{11}$$

However, from the two-dimensional form of Green's formula, if n is the local outward normal to the boundary of the section and s is measured anticlockwise around this boundary,

$$-\int_A \left[\phi_0\left(\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial y^2}\right)-\phi_1\left(\frac{\partial^2\phi_0}{\partial x^2}+\frac{\partial^2\phi_0}{\partial y^2}\right)\right]\mathrm{d}A = -\int_a\left(\phi_0\frac{\partial\phi_1}{\partial n}-\phi_1\frac{\partial\phi_0}{\partial n}\right)\mathrm{d}s = 0. \tag{12}$$

The last equality holds because ϕ_0 and ϕ_1 are zero on the boundary. Then, from (11) and (12), (8) becomes

$$U_S = \frac{1}{2G}\int_A\left[f-\frac{Sx^2}{2I}\right]^2+\phi_0\left[\frac{\partial^2\phi_0}{\partial x^2}+\frac{\partial^2\phi_0}{\partial y^2}\right]-\left(\frac{v}{1+v}\right)^2\phi_1\left[\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial y^2}\right]\mathrm{d}A. \tag{13}$$

It follows from (3) and (10) that f, ϕ_0 and ϕ_1 will be directly proportional to S/I and that none of them is a function of Poisson's ratio. Then, if

$$f = F\frac{S}{I},\quad \phi_0 = \Phi_0\frac{S}{I},\quad \phi_1 = \Phi_1\frac{S}{I}, \tag{14}$$

(13) can be written as

$$U_S = \frac{S^2}{2GA}\left[B+C\left(\frac{v}{1+v}\right)^2\right] = \frac{S^2}{2K_S}\quad\text{or}\quad K_S = \frac{GA}{B+C\left(\frac{v}{1+v}\right)^2} \tag{15}$$

where

$$B = \frac{A}{I^2}\int_A\left[\left(F-\frac{1}{2}x^2\right)^2+\Phi_0\left(\frac{\partial^2\Phi_0}{\partial x^2}+\frac{\partial^2\Phi_0}{\partial y^2}\right)\right]\mathrm{d}A$$

$$C = -\frac{A}{I^2}\int_A\Phi_1\left[\frac{\partial^2\Phi_1}{\partial x^2}+\frac{\partial^2\Phi_1}{\partial y^2}\right]\mathrm{d}A \tag{16}$$

and K_S is the shear stiffness of the section. The conditions governing Φ_0, Φ_1 and F are

$$\frac{\partial^2\Phi_0}{\partial x^2}+\frac{\partial^2\Phi_0}{\partial y^2} = -\frac{\mathrm{d}F}{\mathrm{d}y},\quad \frac{\partial^2\Phi_1}{\partial x^2}+\frac{\partial^2\Phi_1}{\partial y^2} = y \tag{17}$$

within the cross-section and

$$\Phi_0 = \Phi_1 = \left(\frac{1}{2}x^2-F\right)\frac{\mathrm{d}y}{\mathrm{d}s} = 0 \tag{18}$$

on the boundary. For example, for a circular section of radius R,

$$F = \frac{1}{2}(R^2-y^2),\quad -\Phi_0 = \Phi_1 = \frac{1}{8}(R^2-x^2-y^2)y,\quad A = \pi R^2,\quad I = \frac{\pi R^4}{4} \tag{19}$$

giving $B = 7/6$, $C = 1/6$ as found previously.

The general form of K_S given by (15) was inferred in the author's earlier paper (1991) from a number of particular solutions. It was also postulated that B must be greater than (or equal to) unity and that C must be positive. The expression for K_S is valid for the full range of admissible values of v, which lie between the limits of minus one and plus one half. An upper limit to the value of K_S can be found from minimum potential energy principles, as in Rayleigh's method. Applying a kinematically-admissible displacement field to the beam, and comparing the work done by the end loading with the internal strain energy stored yields a beam stiffness which is greater than (or equal to) the true beam stiffness. Prager (1961) uses this method to find an upper bound to the torsional stiffness of a prismatic beam with a square cross-section. In the present case, the work done in shearing is compared with the shear strain energy. Take the shear deformation to be a constant value γ along the beam and over the cross-section. Then, for a beam of length l the work done by the shear force S during shear is $1/2S\gamma l$ and the shear strain energy stored is $1/2G\gamma^2Al$. On comparing the two, an upper limit for K_S is given by S/γ which is equal to GA. Examining the expression for K_S in (15) for the particular case when ν is zero then shows that B cannot be less than unity. As ν tends towards minus one, the term in C becomes dominant in the denominator of this expression. As K_S must be positive, it follows that C cannot be negative.

3. COMPARISON WITH INDEPENDENT RESULTS

Timoshenko's solution has only been applied to simply-connected cross-sections. Love (1952) describes the older Saint-Venant solution for prismatic beams subject to bending and shear in which the state of stress is deduced from a function χ. For a circular tube of inner radius a and outer radius b, χ is given in polar coordinates by

$$\chi = -\left(\frac{3}{4}+\frac{1}{2}\nu\right)\left[(a^2+b^2)r+\frac{a^2b^2}{r}\right]\cos\theta+\frac{1}{4}r^3\cos 3\theta+\text{const.} \tag{20}$$

where r is the radius from the centre and θ is measured anticlockwise from the inwards direction of the shear force. After some manipulation, this yields the shear stresses in polar coordinates

$$\tau_{rz} = \frac{S\cos\theta}{2(1+\nu)I}\left(\frac{3}{4}+\frac{1}{2}\nu\right)\left[r^2-a^2-b^2+\frac{a^2b^2}{r^2}\right]$$

$$\tau_{\theta z} = -\frac{S\sin\theta}{2(1+\nu)I}\left[\left(\frac{3}{4}+\frac{1}{2}\nu\right)\left(a^2+b^2+\frac{a^2b^2}{r^2}\right)+\left(\frac{1}{2}\nu+\frac{1}{4}\right)r^2\right]. \tag{21}$$

As expected, the average tangential shear stress is that predicted from the simple engineering theory. The shear stiffness can be deduced from the shear strain energy and is given by

$$K_S = \frac{6GA(a^2+b^2)^2}{(7a^4+34a^2b^2+7b^4)+(b^2-a^2)^2\left(\frac{\nu}{1+\nu}\right)^2}. \tag{22}$$

This reduces to the solution for a solid circular section as a tends to zero. It also corresponds to the general form for K_S found in the previous section.

Schramm *et al.* (1994) have found beam shear stiffnesses for various shapes of cross-section by numerical methods. Two different types of shear stiffness are found, one by using a geometrical approach and the other using an energy method. The latter is consistent with the generalized beam theory used by the author (1991). Their values for a rectangular section agree with those deduced from the results given in Section 3.1 of the author's paper. They consider the general case of asymmetric cross-sections for which the shear stiffnesses may be coupled. That is, a shear force may produce a shear displacement in a direction at right-angles to it. For consistency, their x, y and z axes will be replaced by the corresponding z, x and y axes, respectively, used here. Then the rates of shear displacement in the x and y directions, γ_x and γ_y, are taken to be related to the shear forces in these directions, S_x and S_y, by the equations

$$\gamma_x = \frac{\alpha_{xx}}{GA}S_x+\frac{\alpha_{xy}}{GA}S_y$$

$$\gamma_y = \frac{\alpha_{yx}}{GA}S_x+\frac{\alpha_{yy}}{GA}S_y \tag{23}$$

where the coefficients α_{xy} and α_{yx} are necessarily equal when the energy approach is used. The α coefficients are tabulated for Poisson's ratio of zero, 0.3 and 0.5. In the absence of coupling terms, the α coefficients should be of the same form as the denominator of the expression for K_S in (15). In particular, they should be equal to B when Poisson's ratio is zero. Apart from the rectangular section, the authors tabulate results for cross-sections in the form of a trapezium, an equal angle and an unequal angle. These are listed in the following table.

Table 1. Shear coefficients derived numerically by Schramm *et al.* and the corresponding coefficients B and C which will generate them

Section	Coefficient	$\nu = 0\ (\equiv B)$	C	$\nu = 0.3$	$\nu = 0.5$
Trapezium	α_{xx}	1.313417	2.853707	1.465389	1.630496
	α_{xy}	−0.063785	−0.230583	−0.076064	−0.089406
	α_{yy}	1.169516	0.022310	1.170704	1.171995
Equal angle	α_{xx}	2.301059	0.035226	2.302935	2.304973
	α_{xy}	0.014969	0.034059	0.016783	0.018753
	α_{yy}	2.301509	0.035226	2.302935	2.304973
Unequal angle	α_{xx}	3.058207	0.066791	3.061764	3.065628
	α_{xy}	0.039510	0.030291	0.041123	0.042876
	α_{yy}	1.898375	0.014328	1.899138	1.899967

The values of B have been chosen to give the correct α coefficients when ν is zero. The chosen values of C then give the correct α coefficients when ν is 0.3 or 0.5. Note that the constraints on the values of B and C do not apply to the coupling coefficients α_{xy}.

4. CONCLUDING REMARKS

The conjecture concerning the form of the shear stiffness coefficient has been proved, using Timoshenko's solution for beams subject to a constant shear force. It has been seen that the same form also applies to cases where Timoshenko's solution may not be appropriate.

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